Decomposition Polyhedra of Piecewise Linear Functions

DISCOGA Seminar

November 8, 2025
Moritz Grillo

Joint work with

Marie-Charlotte Brandenburg (RUB) and Christoph Hertrich (UTN)

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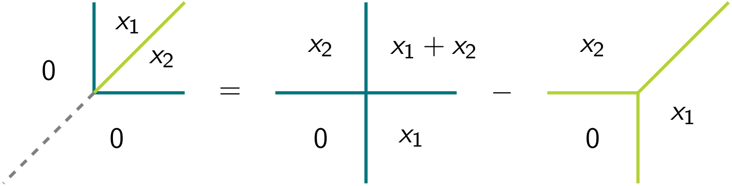
- Nonconvexity complicates optimization and neural network representations.
- Every CPWL function f can be written as difference of two convex CPWL functions $g - h$ [Melzer, 1986; Kripfganz and Schulze, 1987].



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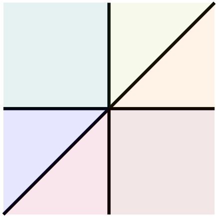
3. Submodular functions

Polyhedral Complex

A **polyhedral complex** $\mathcal{P}$ is a collection of polyhedra such that



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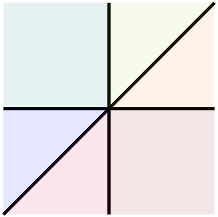

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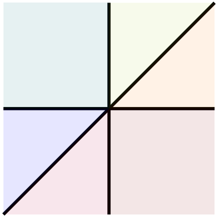

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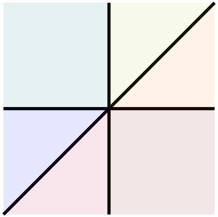

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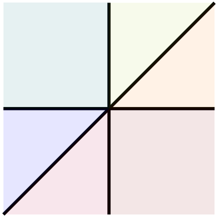

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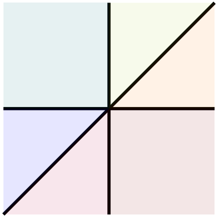

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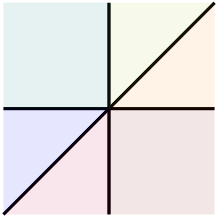

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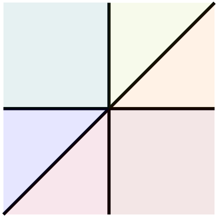

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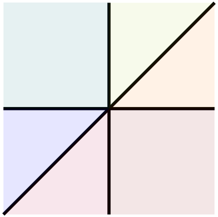

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- If all polyhedra are cones, then we all $\mathcal{P}$ a **polyhedral fan**.

CPWL function

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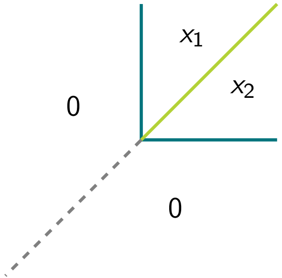

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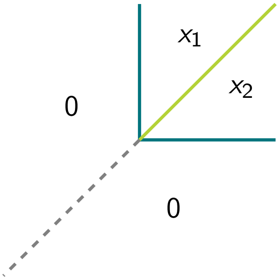

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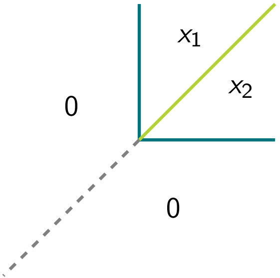

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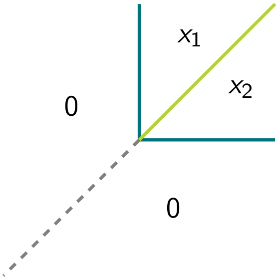
- The **number of pieces** q of f is the smallest possible number of regions $|\mathcal{P}^d|$ of a compatible polyhedral complex $\mathcal{P}$.



Convexity

- If f is a **convex** CPWL function, then

$$f(x) = \max_{i \in [k]} \{\langle a_i, x \rangle + b_i\}$$

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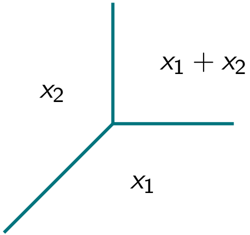
- Example: $\max\{x_1, x_2, x_1 + x_2\}$



Existing Decompositions

Hyperplane extension

Kripfganz and Schulze '87
Zalgaller '00
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- Local maxima decomposition is refinement of the hyperplane extension in the worst case

Hyperplane extension

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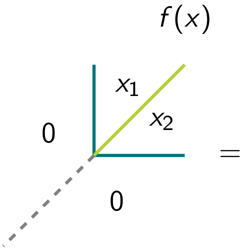

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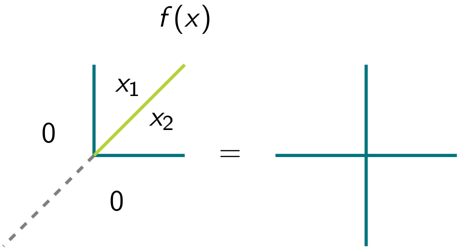

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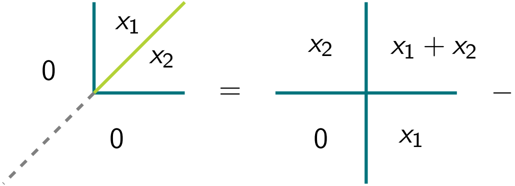

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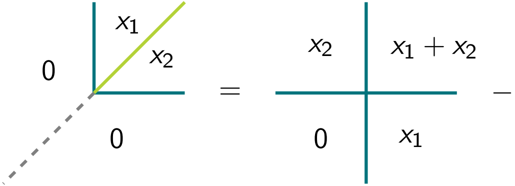

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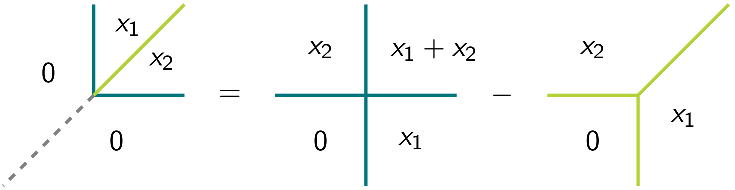

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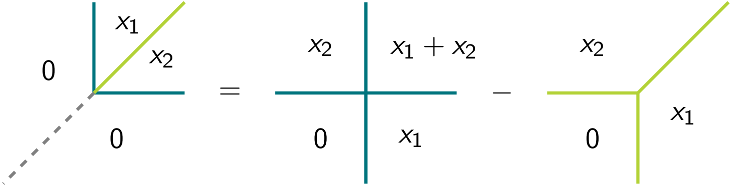
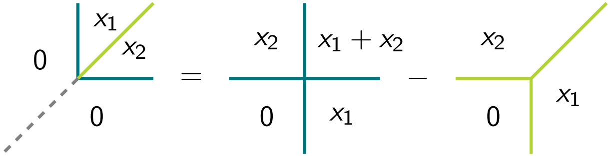


- But: If f has q pieces, g and h can have $O(q^d)$ many pieces

Minimal Decompositions

Is this construction optimal?



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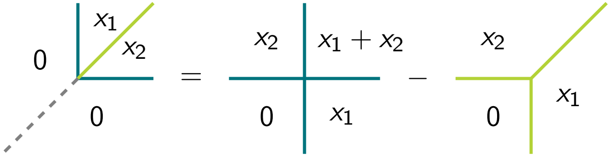

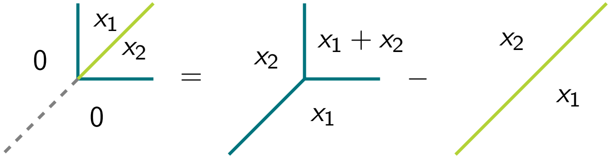

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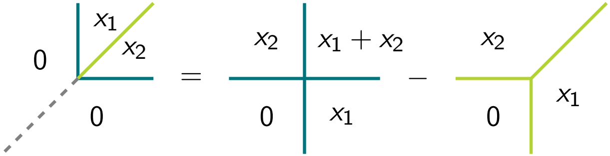

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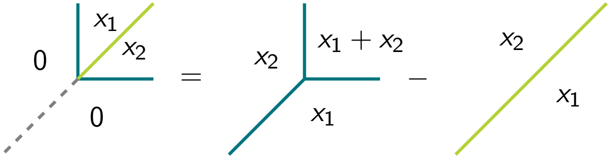

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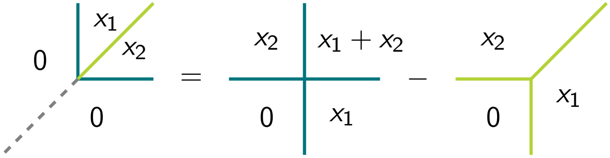

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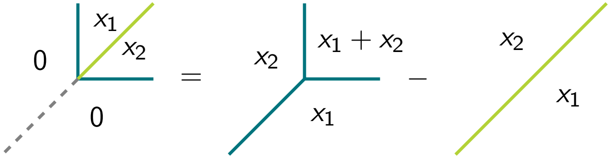

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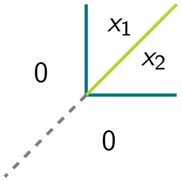

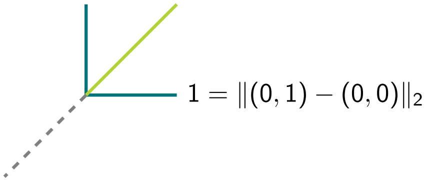

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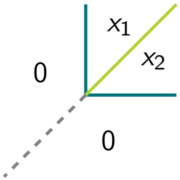

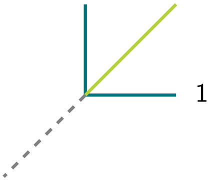

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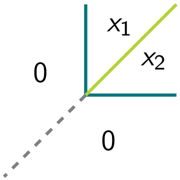

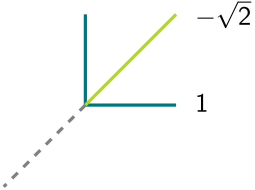

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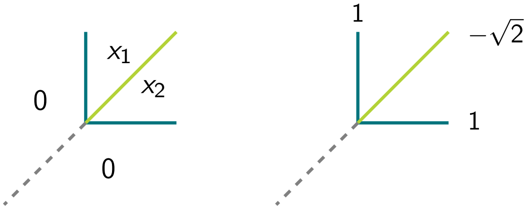

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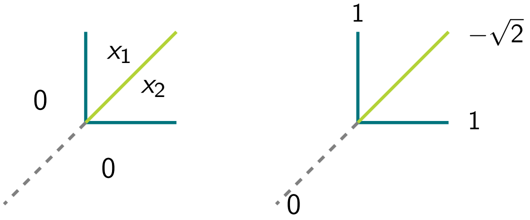

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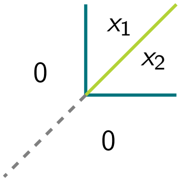

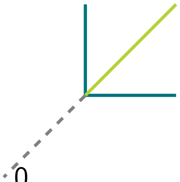

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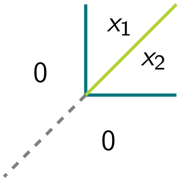
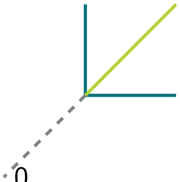
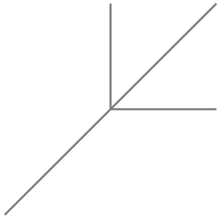




- Then $(\mathcal{P}, w_f)$ is a *balanced complex*.

Balanced polyhedral complexes

- Let $\mathcal{P}$ be polyhedral fan in $\mathbb{R}^2$



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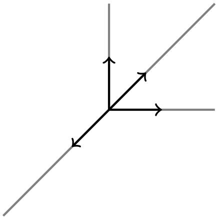

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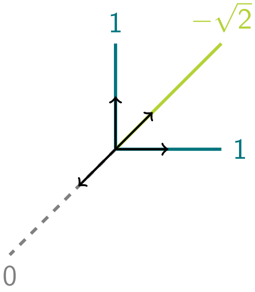
$$\sum_{\sigma \in \mathcal{P}^1} w(\sigma) r_\sigma = 0$$



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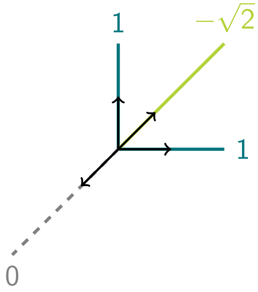

$$\sum_{\sigma \in \mathcal{P}^1} w(\sigma) r_\sigma = 1 \cdot (1,0) - \sqrt{2} \cdot \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) + 1 \cdot (0,1) = (0,0)$$

Balancing condition in higher dimension

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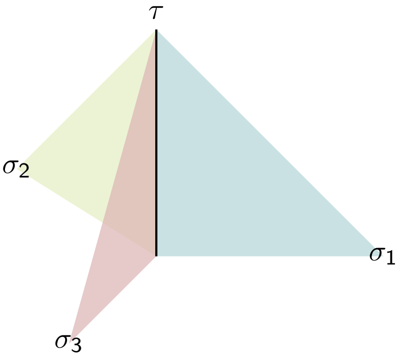

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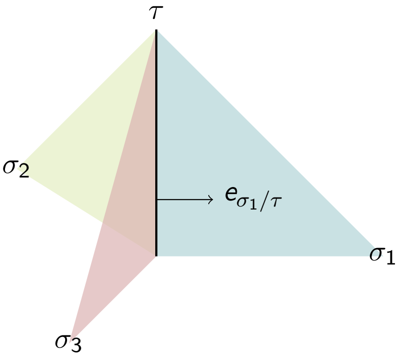

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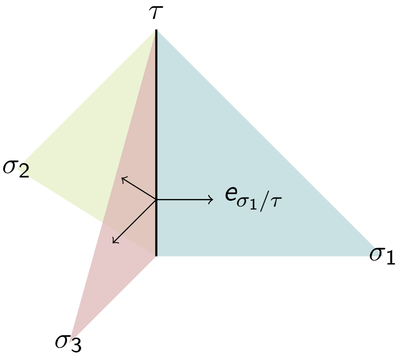

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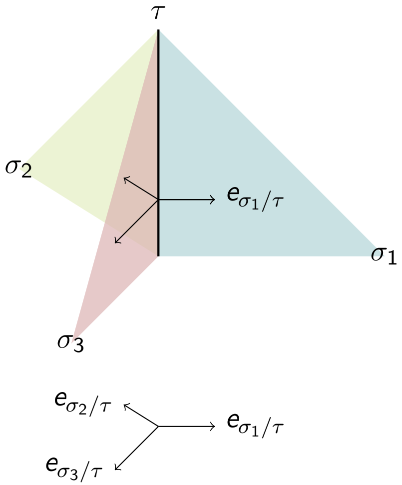
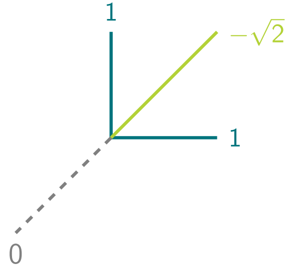


Tropical Geometry

Structure Theorem

- w_f is balanced.

Proof sketch



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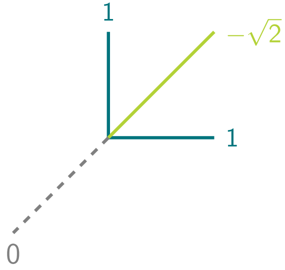

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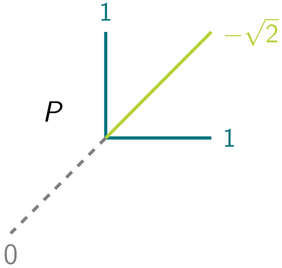

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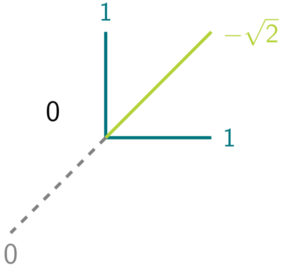

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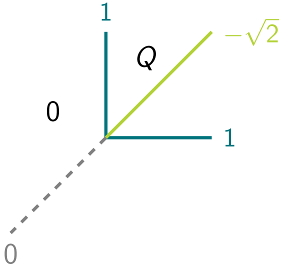

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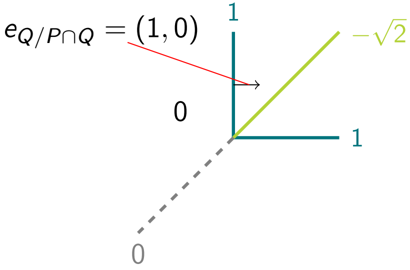

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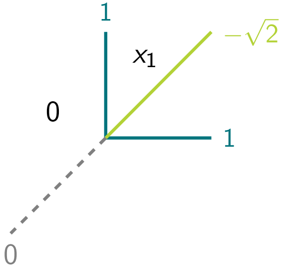

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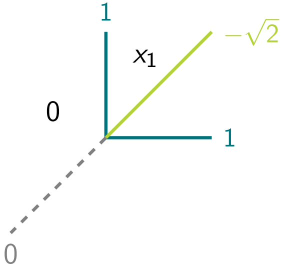

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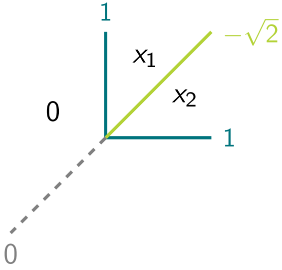

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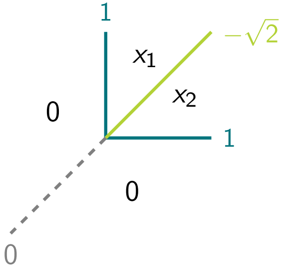

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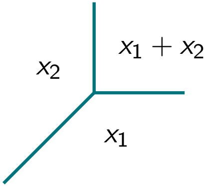

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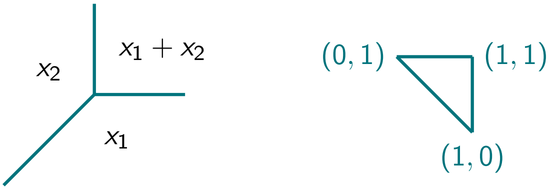

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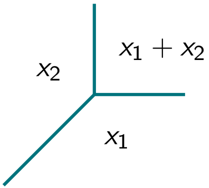

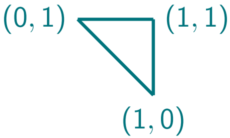

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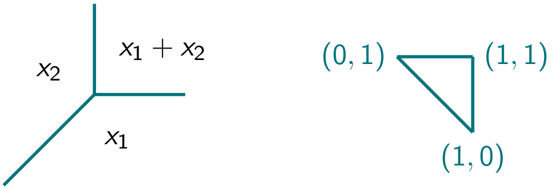

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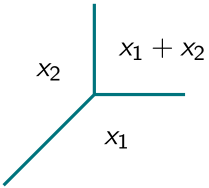

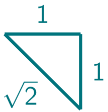

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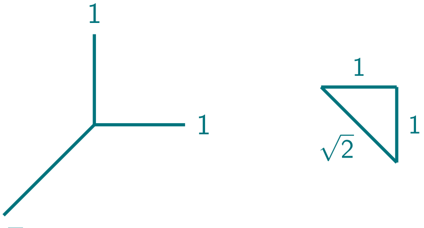
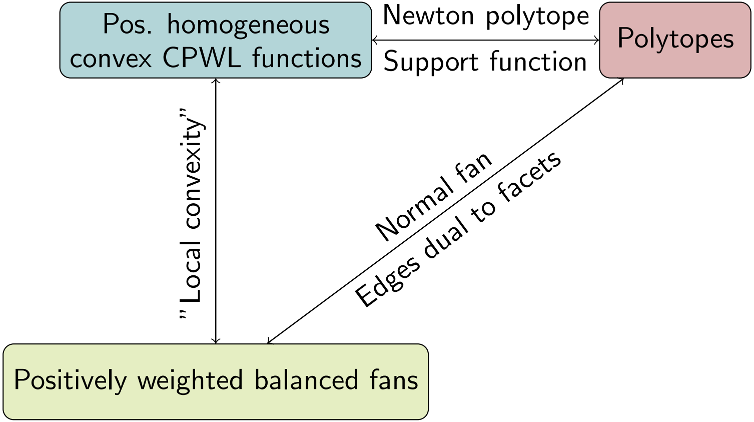


Summary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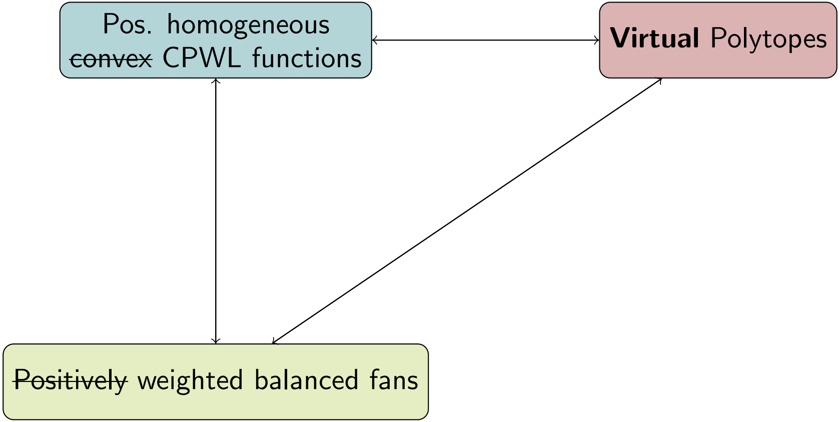
Newton polytope
Support function
Polytopes
"Local convexity"
Normal fan
Edges dual to facets
Positively weighted balanced fans

Allowing Subtraction



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Theorem (Tran and Wang, 2024)

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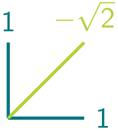

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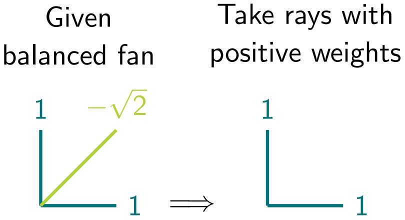

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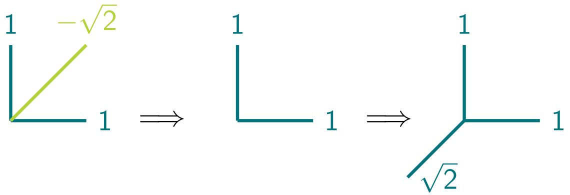

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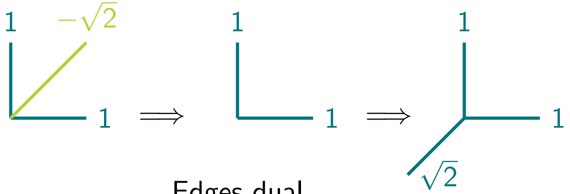

Dual perspective:

Edges dual to rays

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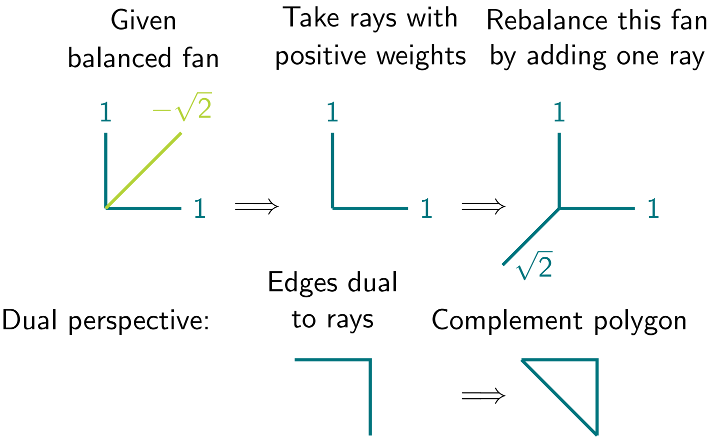

Conjecture for higher dimension

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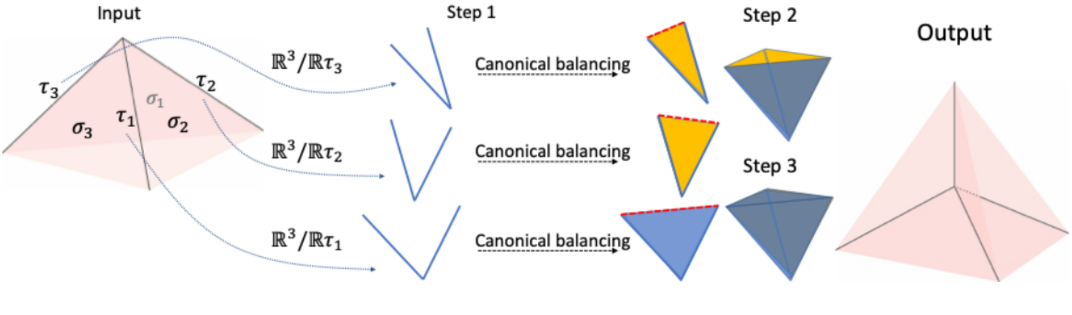

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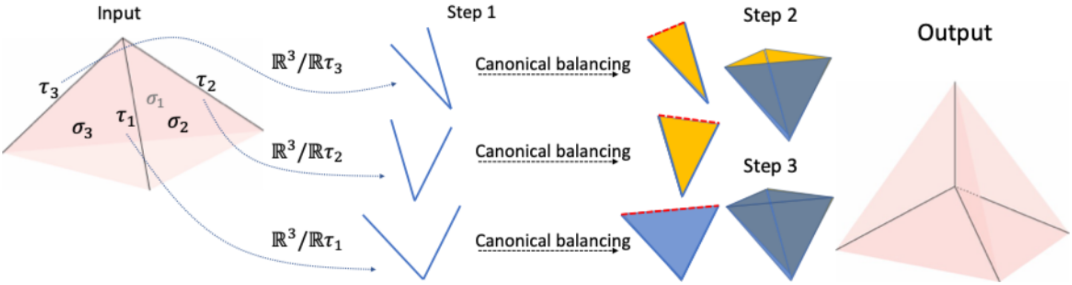

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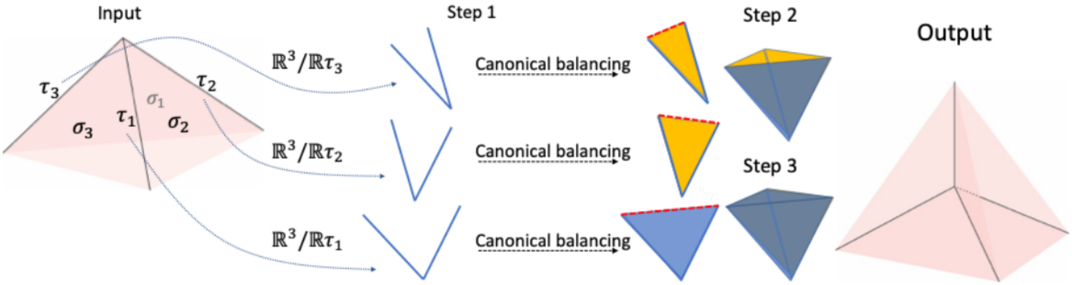

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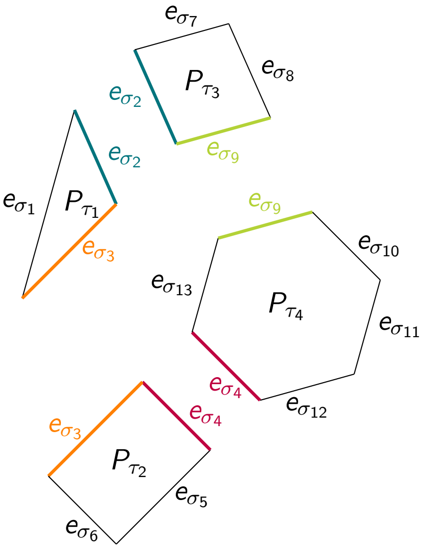
Counterexample to Conjecture



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What is the current state?

- Optimal solution in dimension 2

What now?

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- To make the problem more tractable: Fix the possible breakpoints of the decompositions

Fix the possible locus of nonlinearity

Some observations

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- $\overline{\mathcal{V}_\mathcal{P}}$ is finite dimensional and the set of all convex functions

$$\overline{\mathcal{V}_\mathcal{P}^+} \cong \mathcal{W}_\mathcal{P}^+ := \bigcap_{\sigma \in \mathcal{P}^{d-1}} \{w \in \mathcal{W}_\mathcal{P} \mid w(\sigma) \geq 0\}$$

is a polyhedral cone.

Decomposition Polyhedra

Definition

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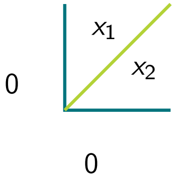

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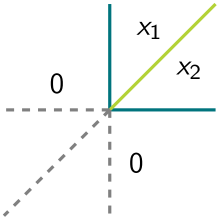

Minimal and Reduced Decompositions

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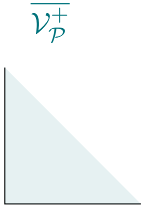
  - and one of the two has strictly fewer pieces.

Structure of the Decomposition Polyhedra

Theorem (Brandenburg, G., Hertrich, 2025)

- *$\mathcal{D}_{\mathcal{P}}(f)$ is a polyhedron that arises as the intersection of two shifted cones.*



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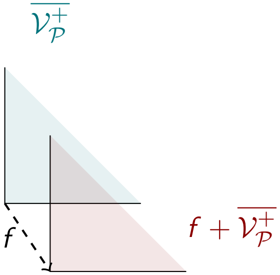

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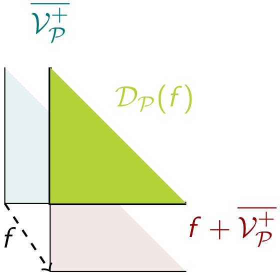

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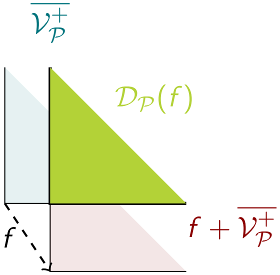

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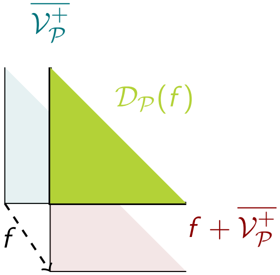

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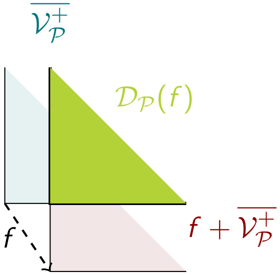
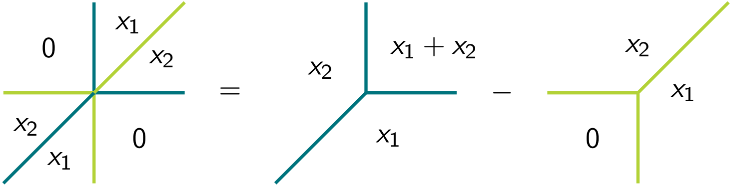


- **Finite procedure** to find minimal decomposition among decompositions in $\mathcal{D}_{\mathcal{P}}(f)$.

Cases with a unique vertex

- Hyperplane functions $f(x) = \sum_{i \in [k]} \lambda_i \cdot \max\{\langle x, a_i \rangle, \langle x, b_i \rangle\}$

- The function $f : \mathbb{R}^n \to \mathbb{R}$ that returns the k-th largest entry of an input vector $x \in \mathbb{R}^n$.



Braid arrangement

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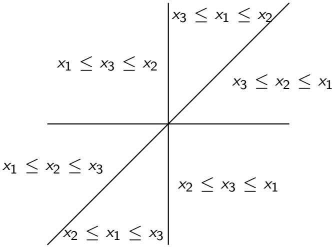

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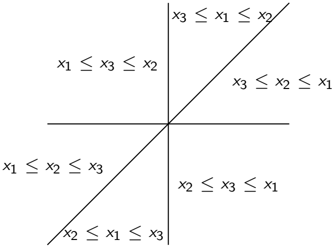

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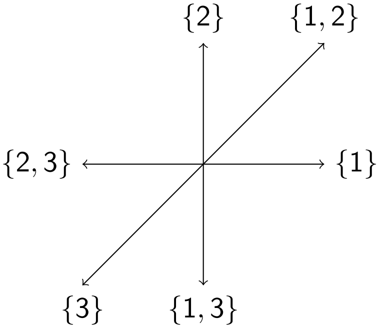

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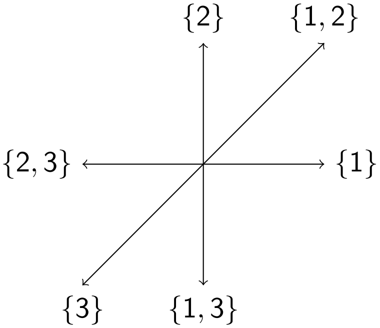

Lemma

The mapping Φ that maps $f \in \mathcal{V}_{\mathcal{B}}$ to the set function $F(S) = f(\mathbb{1}_S)$ is a vector space isomorphism, where $\mathbb{1}_S = \sum_{i \in S} e_i$.

Submodular function

A set function $F\colon 2^{[d]} \to \mathbb{R}$ is called **submodular** if

$$F(A) + F(B) \geq F(A \cup B) + F(A \cap B)$$

for all $A, B \subseteq [d]$ and $F(\emptyset) = 0$.

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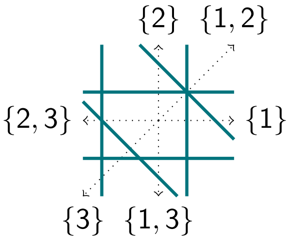

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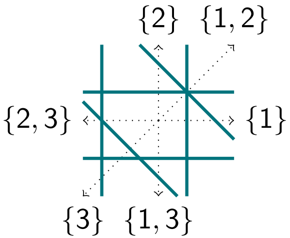
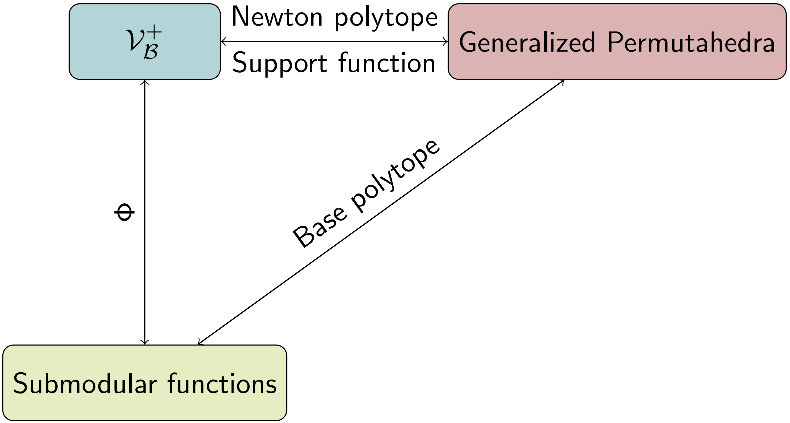


A function f is convex if and only if $\Phi(f)$ is submodular.

Decomposing Set Functions

How to decompose a set function into a difference of submodular functions such that their base polytopes have as few vertices as possible?



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The decomposition polyhedra can be very complex

We do not even know the number of rays of the submodular cone for $d > 5$....

Open Problems (almost everything)

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Thank you for your attention!